International Journal of Theoretical Physics, Vol. 46, No. 8, August 2007 (© 2007)
DOI: 10.1007/s10773-006-9332-8

DKP Equation with Smooth Potential and Position-Dependent Mass

M. Merad[1]

Received July 22, 2006; accepted November 21, 2006
Published Online: January 31, 2007

The Duffin–Kemmer–Petiau (DKP) equation for spin 0 and 1 with smooth potential and position dependent- mass is solved. The solution is given in terms of the Heun function. The step case for potential and mass are deduced as a limiting case. The boundary conditions are also discussed.

KEY WORDS: DKP equation; smooth potential; position-dependent mass.

PACS Numbers: 03.30.+p, 03.65.Pm, 03.65.Ge, 03.65.Db

1. INTRODUCTION

The problem with position-dependent mass is of considerable significance in various areas of physics, citing for instance quantum well and quantum dots (Cassou *et al.*, 2004; Harrison, 2000; Serra and Lipparini, 1997), semiconductor heterostructures (Bastard, 1988; Gora and Williams, 1969), . . . etc. In addition to its practical application, the quantum study of this problem requires some precautions relative to the choice a correct form of the Hamiltonian. Accordingly, several models were presented in which Hamiltonian is hermetic, the Galilean invariance is preserved (Einevoll and Hemmer, 1988; Lévy-Leblond, 1995; Morrow, 1987; Von Roos, 1983), . . . etc. In consequence, the treatment of the Schrodinger equation with position-dependent mass did not cease developing and a good number of articles were published (Alhaidari, 2003; Chetouani *et al.*, 1995, 1998, 1999; de Sousa Dutra, 2003; Dong and Cassou, 2005; Lévy-Leblond, 1992; Mustafa and Mazharimousavi, 2006a,b; Quesne, 2006; Quesne and Tkachuk, 2004; Tanaka, 2006).

However, the relativistic extension of this problem is also of interests and remains unexplored. The major difficulty in this domain is the spin which is a fundamental physical quantity, playing a significant role in the explanation of the

[1] Département de Physique, Centre Universitaire de Oum-El-Bouaghi, 04000 Oum-El-Bouaghi, Algeria; e-mail: meradm@caramail.com

microscopic phenomena and then its effect cannot be neglected. Consequently, a detailed attention is drawn by a number of researchers to elucidate in this case its physical influence. By the same time, this influence is concretized in the development of the exactly soluble models. The analytical solution of these models is then very required since it enables us to explore jointly relativistic and spinor effects. In other way, when the spin value increases the problem becomes more complicated because the number of coupled equations also increases. Some problems were solved within this framework; for example, the Klein–Gordon (KG) equation is exactly solved, with vector and scalar Hulthen-type potential (Domingue-Adame, 1989), with Coulomb like scalar-plus-vector potentials (Chen *et al.*, 2004a), with vector and scalar exponential type potentials (Diao *et al.*, 2004), with scalar and vector Rosen-Morse-type potential (Yi *et al.*, 2004) and with the generalized Hulthen potential (Chen *et al.*, 2004b) and for the Dirac equation, we found the Coulomb problem (Alhaidari, 2004; Mustafa, 2003), the Kepler problem (Vakarchuk, 2005) and a generalized Hulthen potential (Peng *et al.*, 2006).

Besides these fundamental relativistic equations, there is another interesting relativistic one, namely Duffin–Kemmer–Petiau (DKP) equation (Duffin, 1938; Kemmer, 1939; Petiau, 1936). This latter describes jointly the dynamics of the scalar and vectorial particles (spin 0 and 1). It is similar to that of Dirac, where we replace the algebra of the gamma matrices by another algebra noted as beta matrices. These latter last check a known more complicated algebra said DKP algebra which has three irreducible representations: one-dimension trivial representation, five dimension representation associated to spin 0 and ten dimension representation associated to spin 1. In addition, this equation offers a revival in the hope to find a positive density of probability for the particles with spin (Ghose *et al.*, 2001, 2003; Bonin *et al.*, 2006) and also reveals a physical characteristic which is the charge symmetry (Chetouani *et al.*, 2004).

The purpose of this paper is to generalize our previous work (Chetouani *et al.*, 2004) by solving the one dimensional DKP equation for a system with position-dependent mass in interaction with an external potential. For these mass and potential, we consider the smooth shape. This choice of smooth potential (Chetouani *et al.*, 1995, 1998, 1999; Flugge, 1994; Merad *et al.*, 2000; Peng *et al.*, 2006) (respectively, the smooth mass Baskin and Braginsky (1994)) has a physical importance and especially by its limiting case; a step potential (respectively, a step mass). As is well known, this limit plays an important role in various applications. For example, in semiconductors physics, the potentials and effective masses are often modeled by piecewise constant function: step, rectangular barrier, ... etc. (Cassou *et al.*, 2004; Chen *et al.*, 2006; Filikhin *et al.*, 2004; Lévy-Leblond, 1992; Rakityansky, 2004; Shi *et al.*, 1997; Smagley *et al.*, 2002; Zheng *et al.*, 1997).

On the other hand, the naive treatment of the boundary conditions for this step potential (and mass) brings us directly to the trivial solution. This difficulty is avoided by taking the smooth potential (and mass) like starting potential (and mass). Consequently, the exact solution of this smooth potential (and mass) problem permits us to get the generalized adequate boundary conditions.

In Section 2, we expose an explicit calculation relative to the smooth potential and mass in the case of spin 1. We reduce the problem to the Klein–Gordon equation type with an additional mass-dependent term. The exact solution is obtained. The wave functions are expressed by the Heun functions. The reflection and transmission coefficients are evaluated. The limiting cases are considered and the boundary conditions discussed. Following the same method we determine in Section 3 the exact solutions of equation DKP in the case of spin 0 as a particular case.

Before starting the resolution of the DKP equation, let us expose some useful formulas. The one dimension DKP equation interacting with an electromagnetic field and position-dependent mass is given by

$$[i\beta^{\mu}(\partial_{\mu} + ieA_{\mu}(z)) - m(z)]\psi(z, t) = 0, \tag{1}$$

with $\partial_{\mu} = (\partial_0, \partial_3)$, $A_{\mu} = (A_0, A_3)$, $g^{\mu\nu} = diag(+1, -1)$ and where $m(z) = m_0 + S(z)$. The β^{μ} are the DKP matrices and all their properties are listed in Chetouani *et al.* (2004).

It is easy to show that we can obtain the following continuity equation

$$\partial_{\mu} J^{\mu} = 0, \tag{2}$$

where $J^{\mu} \equiv \overline{\psi}\beta^{\mu}\psi$ and the adjoint spinor $\overline{\Psi}$ is defined by

$$\overline{\psi} = \psi^{+}\left(2\left(\beta^{0}\right)^{2} - 1\right). \tag{3}$$

Let us notice that the time-component J^0 of the conserved four-current J^{μ} is not positive definite and may be interpreted as a charge density. It is remarkable to note that this component is positive for positive-energy states and negative for negative-energy ones (Guertin and Wilson, 1977).

In what follows, we are interested in the following choice of

$$A_0(z) = V(z) = \frac{V_0}{2}\left(1 + \tanh\frac{z}{2r}\right), \tag{4}$$

$$A_3(z) = 0, \tag{5}$$

$$m(z) = m_0 + S(z) = m_0 + \frac{S_0}{2}\left(1 + \tanh\frac{z}{2r}\right), \tag{6}$$

where V_0, S_0 and r are suitable dimensional positive parameters.

The step potential is obtained by taking the limit $r \longrightarrow 0$, namely, $\lim\limits_{r\to 0} V(z) \longrightarrow V_0\,\theta(z)$ and $\lim\limits_{r\to 0} m(z) \longrightarrow m_0\,\theta(-z) + (m_0 + S_0)\,\theta(z)$.

The stationary solution of Eq. (1) has the form $\psi(z,t) = e^{-iEt}\phi(z)$ or equivalently we have to solve the following eigenvalue equation

$$\left[\beta^0\left(E - eV(z)\right) + i\beta^3\frac{d}{dz} - (m_0 + S(z))\right]\phi(z) = 0. \tag{7}$$

2. SOLUTION OF THE DKP EQUATION FOR SPIN 1

The DKP equation, as a relativistic equation, is fundamentally related to that of KG one (Lunardi *et al.*, 2000; Nowakowski, 1998). With an aim of converting the form of the problem to that of KG, let us introduce for the system Eq. (7) the following decomposition. We write the wave function $\phi(z)^T = (\varphi, \mathbf{A}, \mathbf{B}, \mathbf{C})$ with $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vectors of dimension (3×1) as

$$\Psi^T = (A_1, A_2, B_3), \quad \Phi^T = (B_1, B_2, A_3), \quad \Theta^T = (C_2, -C_1, \varphi) \text{ and } C_3 \tag{8}$$

where A_i, B_i and C_i, $i = 1, 2, 3$ are respectively the components of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

With these notations, it is not difficult to verify that only the components Ψ are independent and which obey the following Klein–Gordon type equation

$$\left\{(m_0 + S(z))\frac{d}{dz}\left(\frac{1}{(m_0 + S(z))}\frac{d}{dz}\right) + \left[(E - eV(z))^2 - (m_0 + S(z))^2\right]\right\}\Psi = 0. \tag{9}$$

The other components are determined by the following constraint equations

$$\begin{pmatrix}\Phi \\ \Theta\end{pmatrix} = \begin{pmatrix}\dfrac{(E - eV(z))}{m_0 + S(z)} \\ \dfrac{i}{m_0 + S(z)}\dfrac{d}{dz}\end{pmatrix} \otimes \Psi. \tag{10}$$

The component C_3 automatically vanishes ($C_3 = 0$).

Now, in order to solve the Eq. (9), let us introduce the change variable

$$y = \frac{1}{2}\left(1 - \tanh\frac{z}{2r}\right), \tag{11}$$

where y vary in the domain $]0, 1[$. The new form of the Eq. (9) will be written as

$$\begin{aligned}&\frac{1}{r^2}y^2(1-y)^2\frac{d^2\Psi}{dy^2} + \frac{1}{r^2}\left[y(y-1)(2y-1) - \frac{y^2(1-y)^2}{y-a}\right]\frac{d\Psi}{dy}\\ &\quad + \left[E + m_0 + (eV_0 - S_0)(y-1)\right]\left[E - m_0 + (eV_0 + S_0)(y-1)\right]\Psi = 0,\end{aligned} \tag{12}$$

where $a = \frac{m_0 + S_0}{S_0}$.

This last equation can change to the following form

$$\frac{d^2\Psi}{dy^2} + \left[\frac{1}{y} + \frac{1}{y-1} - \frac{1}{y-a}\right]\frac{d\Psi}{dy} + \frac{1}{y(y-1)(y-a)} \times \left[-\omega^2 y + \nu^2 - \mu^2 + a\omega^2 - \frac{a\nu^2}{y} + \frac{(a-1)\,\mu^2}{y-1}\right]\Psi = 0, \quad (13)$$

with the following abbreviations

$$\nu^2 = r^2[(m_0 + S_0)^2 - (E - eV_0)^2],$$
$$\mu^2 = r^2\left(m_0^2 - E^2\right), \quad \text{and } \omega^2 = r^2\left(S_0^2 - (eV_0)^2\right). \quad (14)$$

We note that this equation possesses singular points $y = 0, 1, a, \infty$.

By means of the substitution $\Psi = y^\nu(1-y)^\mu \widetilde{\Psi}$ this equation is reduced to a Heun type equation (Erdélyi *et al.*, 1955)

$$\frac{d^2\widetilde{\Psi}}{dy^2} + \left[\frac{2\nu+1}{y} + \frac{2\mu+1}{y-1} - \frac{1}{y-a}\right]\frac{d\widetilde{\Psi}}{dy} + \frac{1}{y(y-1)(y-a)} \times [b + [(\mu+\nu)^2 - \omega^2]y]\widetilde{\Psi} = 0, \quad (15)$$

with $b = a[\omega^2 - (\mu+\nu)(\mu+\nu+1)] + \nu$.

The regular solution at origin $y = 0$ of this differential equation is

$$\Psi(y) = y^\nu(y-1)^\mu \; H(a, b; \alpha, \beta, \gamma, \delta; y)\mathbf{V}, \quad (16)$$

whose parameters are given by

$$\begin{cases} \alpha = \omega + \mu + \upsilon \\ \beta = -\omega + \mu + \upsilon \\ \gamma = 2\nu + 1 \\ \delta = -1 \end{cases} \quad (17)$$

where $\mathbf{V}$ is a 3 dimension constant vector and $H(a, b; \alpha, \beta, \gamma, \delta; y)$ is the Heun function defined par the series

$$H(a, b; \alpha, \beta, \gamma, \delta; y) = \left\{1 - \frac{b}{\gamma a}y + \sum_{s=2}^{+\infty} c_s y^s\right\} \quad (18)$$

where the c_s coefficients of the series are determined by the difference equation

$$(s+2)(s+1+\gamma)ac_{s+2} = \{(s+1)^2(a+1) + (s+1)[\gamma + \delta - 1 + (\alpha + \beta - \gamma)a] - b\}c_{s+1} - (s+\alpha)(s+\beta)c_s \quad (19)$$

with the initial conditions $c_0 = 1$, $c_1 = -\frac{b}{\gamma a}$ and $c_s = 0$ if $s < 0$.

At this level, the components of Φ^T result directly. For those of Θ^T, we use the following Heun property

$$(\beta-\gamma-\delta)H(a, b+1-a+(a-1)\beta+a\alpha-a\delta;\alpha,\beta-1,\gamma,\delta+1;y)$$
$$=(\alpha y+\beta-\gamma-\delta)H(a,b;\alpha,\beta,\gamma,\delta;y)+y(y-1)\frac{d\ H(a,b;\alpha,\beta,\gamma,\delta;y)}{dy} \tag{20}$$

By a straightforward calculation, it is easy to obtain the following final solution

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix} = y^{\nu}(y-1)^{\mu}$$
$$\times\left[H(a,b;\alpha,\beta,\gamma,\delta;y)\mathbf{M}(y)+H(a,b+1-a+(a-1)\beta\right.$$
$$\left.+a\alpha-a\delta;\alpha,\beta-1,\gamma,\delta+1;y)\mathbf{N}(y)\right], \tag{21}$$

with $\mathbf{M}(y)$ and $\mathbf{N}(y)$ are nine component vectors defined as

$$\mathbf{M}(y)=\begin{pmatrix}1\\ \dfrac{E+eV_0(y-1)}{m_0-eS_0(y-1)}\\ \dfrac{-i\,[\omega(y-1)+\mu]}{r\,(m_0-eS_0(y-1))}\end{pmatrix}\otimes\mathbf{V}\qquad \text{and}$$
$$\mathbf{N}(y)=\begin{pmatrix}0\\ 0\\ \dfrac{-i\,[\omega-\mu+\nu]}{r\,(m_0-eS_0(y-1))}\end{pmatrix}\otimes\mathbf{V}. \tag{22}$$

In fact, the components of the vector $\mathbf{V}(i=1,2,3,)$ are the constants relative to the three directions of the spin 1. Now, by returning to the old variable z (11) and (21), we obtain the result

$$\begin{pmatrix}\Psi\\ \Phi\\ \Theta\end{pmatrix}=\left[\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right]^{\nu}\left[\frac{1}{2}\left(1+\tanh\frac{z}{2r}\right)\right]^{\mu}$$
$$\times\ \exp(i\pi\mu)\left[H\left(a,b;\alpha,\beta,\gamma,\delta;\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right)\mathbf{M}(z)\right.$$
$$+H\left(a,b+1-a+(a-1)\beta+a\alpha-a\delta;\alpha,\beta-1,\gamma,\delta\right.$$

$$+\ 1; \frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\Big)\mathbf{N}(z)\Big], \tag{23}$$

with

$$\mathbf{M}(z) = \begin{pmatrix} 1 \\ \dfrac{2E - eV_0(1+\tanh\frac{z}{2r})}{2m_0 + S_0(1+\tanh\frac{z}{2r})} \\ \dfrac{i\omega(1+\tanh\frac{z}{2r}) + 2\mu}{r\left[2m_0 + S_0(1+\tanh\frac{z}{2r})\right]} \end{pmatrix} \otimes \mathbf{V} \tag{24}$$

$$\mathbf{N}(z) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{-2i\,(\omega - \mu + \nu)}{r\left[2m_0 + S_0(1+\tanh\frac{z}{2r})\right]} \end{pmatrix} \otimes \mathbf{V}. \tag{25}$$

Let us now pass on to a discussion of behavior of the wave function of smooth potential and mass at $\pm\infty$. First, when $z \to -\infty$ (or $y \to 1$), we have the limits

$$\lim_{y\to 1} y^{\nu} \longrightarrow 1, \quad \lim_{y\to 1}(y-1)^{\mu} \longrightarrow \exp(i\pi\mu)\exp(\mu z/r), \tag{26}$$

we use the property of the Heun function which links the y and $1-y$ argument,

$$\begin{aligned} H(a,b;\alpha,\beta,\gamma,\delta;y) = {} & D_1.H(1-a,-b-\alpha\beta;\alpha,\beta,1+\alpha \\ & +\beta-\gamma-\delta,\delta;1-y) + D_2.(1-y)^{\gamma+\delta-\alpha-\beta} \\ & \times H(1-a,-b-\alpha\beta-(\gamma+\delta-\alpha-\beta)(\gamma+\delta-a\gamma);\gamma \\ & +\delta-\alpha,\gamma+\delta-\beta,1+\gamma+\delta-\alpha-\beta,\delta;1-y) \end{aligned} \tag{27}$$

with constants D_1 and D_2 are

$$\begin{aligned} D_1 = H(a,b;\alpha,\beta,\gamma,\delta;1), \quad & D_2 = H(a,b-\alpha\gamma\left[\gamma+\delta-\alpha-\beta\right]; \\ & \gamma+\delta-\alpha,\gamma+\delta-\beta,\gamma,\delta;1) \end{aligned} \tag{28}$$

and $H(a,b;\alpha,\beta,\gamma,\delta;0) \longrightarrow 1$, we get the following behavior of (23) for $z \to -\infty$

$$\begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}_{z\to-\infty} \to \exp(i\pi\mu)\left[\begin{pmatrix} D_1.e^{\mu z/r} + D_2.e^{-\mu z/r} \\ \dfrac{E}{m_0}\left(D_1.e^{\mu z/r} + D_2.e^{-\mu z/r}\right) \\ \dfrac{i\mu}{m_0 r}\left(D_1 e^{\mu z/r} - D_2.e^{-\mu z/r}\right) \end{pmatrix} \otimes \mathbf{V}\right]. \tag{29}$$

The value of μ in (14) is always an imaginary parameter with positive sign in the interval $m_0 < E < +\infty$.

Setting $\mu = irk_1$, with $k_1^2 = E^2 - m_0^2$ where k_1 is real positive

$$\begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}_{z\to-\infty} \to \exp(-\pi r k_1) \left[D_1 \begin{pmatrix} 1 \\ \dfrac{E}{m_0} \\ \dfrac{-k_1}{m_0} \end{pmatrix} .e^{ik_1 z} + D_2 \begin{pmatrix} 1 \\ \dfrac{E}{m_0} \\ \dfrac{k_1}{m_0} \end{pmatrix} .e^{-ik_1 z} \right] \otimes \mathbf{V}. \tag{30}$$

We notice that this expression composed of an incident wave being propagated from $-\infty$ and another reflected wave being propagated to $-\infty$ conforming to the physical problem.

For $z \to +\infty$ (or $y \to 0$), we use the limits

$$\lim_{y\to 0} y^{\nu} \longrightarrow e^{-\nu z/r}, \quad \lim_{y\to 0}(y-1)^{\mu} \longrightarrow \exp(i\pi\mu) \text{ and}$$
$$\lim_{y\to 0} H(a,b;\alpha,\beta,\gamma,\delta;y) \longrightarrow 1. \tag{31}$$

Then the wave function (23) has the following behavior

$$\begin{pmatrix} \Psi \\ \Phi \\ \Theta \end{pmatrix}_{z\to+\infty} \to \exp(i\pi\mu) \left[\begin{pmatrix} 1 \\ \dfrac{E - eV_0}{m_0 + S_0} \\ \dfrac{-i\nu}{r(m_0+S_0)} \end{pmatrix} \exp(-\nu z/r) \right] \otimes \mathbf{V}. \tag{32}$$

The value of ν change according to values of energy in the following intervals:

1) For $eV_0 + (m_0 + S_0) < E < +\infty$, ν becomes purely imaginary with negative sign.

   By setting $\nu = -irk_2$, with $k_2^2 = (E - eV_0)^2 - (m_0 + S_0)^2$ where k_2 is real positive, we obtain a transmitted wave being propagated towards $+\infty$.

   - If $eV_0 > 2m_0 + S_0$:

2) For $m_0 < E < eV_0 - (m_0 + S_0)$, ν becomes purely imaginary with positive sign.

   By setting $\nu = irk_2$, we obtain a transmitted wave being propagated towards $-\infty$ (change of the sign of the wave vector k_2).

3) For $eV_0 - (m_0 + S_0) < E < eV_0 + (m_0 + S_0)$, ν becomes real with positive sign.

   By setting $\nu = rK$ with $K^2 = (m_0 + S_0)^2 - (E - eV_0)^2$ where K is real positive, we obtain an evanescent wave which decays exponentially.

- If $eV_0 < 2m_0 + S_0$ and for $m_0 < E < eV_0 + (m_0 + S_0)$, ν is real with positive sign, we obtain also an evanescent wave.

We note here that for the choices of the signs which we have adopted follow the convergence of the wave functions at $\pm\infty$ and the pairs creation phenomena. For example, in the case, where $\nu = irk_2$, we take account of the discussion which follows the appearance of the Klein paradox by supposing that the field is strong enough at the point ($z = 0$). At this point, we have creation of particle-antiparticle pairs. All incident and created particles move towards the left while antiparticles move towards the right. Consequently, the conservation equation is violated and implies that the reflection coefficient is higher than one (Durand, 1976).

Now, let us calculate the reflection and transmission coefficients along the direction of the spin. Using the definition of the conserved quadrivector (2), in the domain where, $\nu = \mp irk_2$ with $\mu = irk_1$, we find the expression of R as

$$\begin{aligned} R &= \frac{\mid J_{ref} \mid}{\mid J_{inc} \mid} \\ &= \left| \frac{H(a, b - \alpha\gamma\left[\gamma + \delta - \alpha - \beta\right], \gamma + \delta - \alpha, \gamma + \delta - \beta, \gamma, \delta; 1)}{H(a, b; \alpha, \beta, \gamma, \delta; 1)} \right|^2 . \end{aligned} \tag{33}$$

and T

$$\begin{aligned} T &= \frac{\mid J_{tr} \mid}{\mid J_{inc} \mid} \\ &= \frac{(a-1)}{a} \frac{|\nu - \nu^*| \left|\exp\left[-\frac{z}{r}(\nu + \nu^*)\right]\right|}{|2\mu|} \frac{1}{|H(a, b; \alpha, \beta, \gamma, \delta; 1)|^2}. \end{aligned} \tag{34}$$

But in the domain where $\nu = rK$ is real, the value of T vanishes ($T = 0$), Eq. (32) describes total reflection of the incident wave so that the coefficient of reflection must become equal one.

Now it is interesting to study the following particular cases:

a) Smooth potential and null scalar potential ($a \longrightarrow \infty$ or $S_0 \longrightarrow 0$)

For this purpose, we use the following lim it (Snow, 1952), which gives one of the case where the Heun function degenerates into a hypergeometric function

$$\lim_{a \to \infty} H(a, al; \alpha', \beta', \gamma', \delta'; z) \longrightarrow {}_2F_1\left(\zeta + \sqrt{\zeta^2 + l}, \zeta - \sqrt{\zeta^2 + l}; \gamma'; z\right), \tag{35}$$

with

$$\zeta = \frac{\alpha' + \beta' - \delta'}{2}.$$

Applying this limit (35) for the Heun function contained in the expressions of the reflection and transmission coefficients (33, 34), we find

$$\lim_{a\to\infty} R \longrightarrow$$
$$\left|\frac{{}_2F_1(\widetilde{\upsilon}-\mu+\frac{1}{2}+\sqrt{\frac{1}{4}-e^2V_0^2r^2},\widetilde{\upsilon}-\mu+\frac{1}{2}-\sqrt{\frac{1}{4}-e^2V_0^2r^2},2\widetilde{\upsilon}+1;1)}{{}_2F_1(\widetilde{\upsilon}+\mu+\frac{1}{2}+\sqrt{\frac{1}{4}-e^2V_0^2r^2},\widetilde{\upsilon}+\mu+\frac{1}{2}-\sqrt{\frac{1}{4}-e^2V_0^2r^2},2\widetilde{\upsilon}+1;1)}\right|^2 \tag{36}$$

$$\lim_{a\to\infty} T \longrightarrow$$
$$\frac{|\widetilde{\upsilon}-\widetilde{\upsilon}^*|\left|\exp\left[-\frac{z}{r}(\widetilde{\upsilon}+\widetilde{\upsilon}^*)\right]\right|}{|2\mu|\left|{}_2F_1(\widetilde{\upsilon}+\mu+\frac{1}{2}+\sqrt{\frac{1}{4}-e^2V_0^2r^2},\widetilde{\upsilon}+\mu+\frac{1}{2}-\sqrt{\frac{1}{4}-e^2V_0^2r^2},2\widetilde{\upsilon}+1;1)\right|^2} \tag{37}$$

where $\widetilde{\upsilon}^2 = r^2[m_0^2-(E-eV_0)^2]$, which are exactly the same result obtained in Chetouani *et al.* (2004).

b) Step potential and null scalar potential ($r \longrightarrow 0$ and $a \longrightarrow \infty$)

We consider the limiting case when the smooth potential tends to step potential and we retain the constant mass, the limit of the reflection and transmission coefficients can be deduced by taking account of the properties of a hypergeometric function (Gradshtein and Ryzhik, 1965)

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad \text{and} \quad z\Gamma(z) = \Gamma(1+z) \tag{38}$$

as follows

For $E > eV_0+m_0$, $\widetilde{\upsilon} = -ir\widetilde{k}_2$ is imaginary, with $\widetilde{k}_2^2 = (E-eV_0)^2-m_0^2$ where $\widetilde{k}_2$ is real positive

$$R = \frac{(k_1-\widetilde{k}_2)^2}{(k_1+\widetilde{k}_2)^2}, \quad T = \frac{4k_1\widetilde{k}_2}{(k_1+\widetilde{k}_2)^2}, \quad \text{and} \quad R+T=1. \tag{39}$$

- If $eV_0 > 2m_0+S_0$ and for $eV_0-m_0 < E < eV_0+m_0$, $\nu = r\widetilde{K}$ is real with $\widetilde{K}^2 = m_0^2-(E-eV_0)^2$ where $\widetilde{K}$ is real positive.

$$R = 1 \quad \text{and} \quad T = 0. \tag{40}$$

- If $eV_0 > 2m_0 + S_0$ and for $m_0 < E < eV_0 - m_0$, $\widetilde{\upsilon} = ir\widetilde{k}_2$ is imaginary with positive sign.

$$R = \frac{(k_1 + \widetilde{k}_2)^2}{(k_1 - \widetilde{k}_2)^2}, \quad T = \frac{4k_1\widetilde{k}_2}{(k_1 - \widetilde{k}_2)^2}, \quad \text{and} \quad R - T = 1. \tag{41}$$

We notice that $R > 1$, this anomaly is restored by the introduction of the pair creation, which is the Klein's Paradox.

Let us now determine the appropriate boundary conditions for the potential and mass variable admitting a jump at an unspecified point z_0. As it has been said previously, the naive conditions of continuity lead directly to the trivial solution (Chetouani *et al.*, 2004). To find the adequate conditions, we proceed in the following way. Let us start from Ψ which satisfies the Klein–Gordon type modified Eq. (9). Then, we must impose on it and on its derivative the continuity conditions. By integrating the Eq. (9) in the domain $[z_0^-, z_0^+]$, one gets

$$\begin{gathered} \Psi(z_0^+) = \Psi(z_0^-), \\ \frac{1}{m_0 + S(z_0^+)} \frac{d\Psi(z_0^+)}{dz} = \frac{1}{m_0 + S(z_0^-)} \frac{d\Psi(z_0^-)}{dz} \end{gathered} \tag{42}$$

Using these conditions we obtain

$$\begin{pmatrix} \Psi(z_0^+) \\ \Phi(z_0^+) \\ \Theta(z_0^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{(E - eV_2(z_0^+))(m_0 + S(z_0^-))}{(E - eV_1(z_0^-))(m_0 + S(z_0^+))} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi(z_0^-) \\ \Phi(z_0^-) \\ \Theta(z_0^-) \end{pmatrix} \tag{43}$$

It is not difficult to check that in the case of step potential and mass conditions are satisfied.

3. SOLUTION OF THE DKP EQUATION FOR SPIN 0

Let us proceed in the same way as in the case of spin 1, $\phi(z)^T = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$, the system Eq. (7) is reduced to the following system

$$\begin{cases} \left\{(m_0 + S(z)) \dfrac{d}{dz}\left(\dfrac{1}{m_0 + S(z)} \dfrac{d}{dz}\right) + \left[(E - eV(z))^2 - (m_0 + S(z))^2\right]\right\} \eta_1 = 0 \\ \eta_2 = \dfrac{(E - eV)}{(m_0 + S(z))}\eta_1 \\ \eta_3 = 0 \\ \eta_4 = 0 \\ \eta_5 = \dfrac{i}{(m_0 + S(z))} \dfrac{d\eta_1}{dz}, \end{cases} \tag{44}$$

Following the same stages as in the preceding case, we arrive at the result

$$\phi(z) = C\left[\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right]^{\nu}\left[\frac{1}{2}\left(1+\tanh\frac{z}{2r}\right)\right]^{\mu}\frac{\exp(i\pi\mu)}{\left[2m_0+S_0\left(1+\tanh\frac{z}{2r}\right)\right]}$$

$$\begin{pmatrix} \left[2m_0+S_0\left(1+\tanh\frac{z}{2r}\right)\right]H\left(a,b;\alpha,\beta,\gamma,\delta;\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right) \\ \left[2E-eV_0\left(1+\tanh\frac{z}{2r}\right)\right]H\left(a,b;\alpha,\beta,\gamma,\delta;\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right) \\ 0 \\ 0 \\ \frac{\left[i\omega(1+\tanh\frac{z}{2r})+2\mu\right]}{r}H\left(a,b;\alpha,\beta,\gamma,\delta;\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right) \\ -\frac{2i\,(\omega-\mu+\nu)}{r}H\left(a,\widetilde{b},\alpha,\widetilde{\beta},\gamma,\widetilde{\delta},\frac{1}{2}\left(1-\tanh\frac{z}{2r}\right)\right) \end{pmatrix}. \quad (45)$$

where $\widetilde{b} = b+1-a+(a-1)\beta+a\alpha-a\delta$, $\widetilde{\beta} = \beta-1$ and $\widetilde{\delta} = \delta+1$.

It is noticeable that we obtain the same expressions for R and T respectively given by (33) and (34).

The step potential and mass limit ($r \longrightarrow 0$) is

$$\phi(z) \to C.\exp(i\pi\mu)$$
$$\times\left\{\theta(-z)\left[\begin{pmatrix}1\\ \frac{E}{m_0}\\ 0\\ 0\\ \frac{-k_1}{m_0}\end{pmatrix}D_1.e^{ik_1z}+\begin{pmatrix}1\\ \frac{E}{m_0}\\ 0\\ 0\\ \frac{k_1}{m_0}\end{pmatrix}D_2.e^{-ik_1z}\right]+\theta(z)\begin{pmatrix}1\\ \frac{E-eV_0}{m_0+S_0}\\ 0\\ 0\\ \frac{-k_2}{m_0+S_0}\end{pmatrix}e^{ik_2z}\right\} \quad (46)$$

with the same parameters μ, ν defined in (14).

4. CONCLUSION

We have solved the DKP equation (spin 0 and 1) with smooth potential and mass. The DKP equation was reduced to Klein–Gordon type equation. The resolution of the equation required the use of the Heun functions. The wave functions, the reflection and transmission coefficients are then exactly evaluated.

The generalized boundary conditions are deduced from the smooth potential (and mass) study. The propagation through a jump of potential resembles the case of the propagation of the photon through different mediums. This fact does not surprise since the photon has a spin 1. The limiting cases are then deduced. The Klein paradox is analyzed and it persists. Its solution found within the framework of quantum field theory. This latter is due to the equivalence connecting the DKP and KG formalisms.

ACKNOWLEDGMENTS

The author wish to thank the referees for their helpful comments and suggestions.

REFERENCES

Alhaidari, A. D. (2003). *International Journal of Theoretical Physics* **42**.

Alhaidari, A. D. (2004). *Physics Letters A* **322**, 72.

Baskin, E. M. and Braginsky, L. S. (1994). *Physical Review B (Condensed Matter)* **50**, 12191.

Bastard, G. (1988). *Wave Mechanics Applied to Semiconductor Heterostructure*, Les Editions de Physique, Les Ulis, France.

Bonin, C. A., Lunardi, J. T., Manzoni, L. A., and Pimentel, B.M. (2006). quant-ph/0608002.

Cassou, M. L., Dong, S. H., and Yua, J. (2004). *Physics Letters A* **331**, 45.

Chen, C. Y., Sun, D. S., and Lu, F. L. (2004a). *Physics Letters A* **330**, 424.

Chen, G., Chen, Z. D., and Louy, Z. D. (2004b). *Physics Letters A* **331**, 374.

Chen, X., Li, C. F., and Ban, Y. (2006). *Physics Letters A* **354**, 161.

Chetouani, L., Dekar, L., and Hammann, T. F. (1995). *Physical Review A (Atomic, Molecular, and Optical Physics)* **52**, 82.

Chetouani, L., Dekar, L., and Hammann, T. F. (1998). *Journal of Mathematical Physics* **39**, 2551.

Chetouani, L., Dekar, L., and Hammann, T. F. (1999). *Physical Review A (Atomic, Molecular, and Optical Physics)* **59**, 1.

Chetouani, L., Merad, M., Boudjedaa, T., and Lecheheb, A. (2004). *International Journal of Theoretical Physics* **43**, 1147.

de Sousa Dutra, A., Hott, M., and Almeida, C. A. S. (2003). *Europhysics Letters* **62**, 8.

de Souza Dutra, A. (2006). *Journal of Physics A (Mathematical and General)* **39**, 203.

Diao, Y. F., Yi, L. Z., and Jia, C. S. (2004). *Physics Letters A* **332**, 157.

Domingue-Adame, F. (1989). *Physics Letters A* **136**, 175.

Dong, S. H. and Cassou, M. L. (2005). *Physics Letters A* **337**, 313.

Duffin, R.Y. (1938). *Physical Review* **54**, 1114.

Durand, E. (1976). *Mecanique Quantique, Particule dans un Champ, Tome II: Spin et Relativité*, Ed. Masson, Paris.

Einevoll, G. T. and Hemmer, P. C. (1988). *Journal of Physics C (Solid State Physics)* **21**, L1193.

Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1955). *Higher Transcendental Functions*, Vol. III, McGraw-Hill, New York.

Filikhin, I., Deynekal, E., and Vlahovic, B. (2004). *Modelling and Simulation in Materials Science and Engineering* **12**, 1121.

Flugge, S. (1994). *Pratical Quantum Mechanics, 2nd printing*, Vol. II, Springer, Berlin, p. 213.

Ghose, P., Majumdar, A. S., Guha, S., and Sau, J. (2001). *Physics Letters A* **290**, 205.

Ghose, P., Samal, M. K., and Datta, A. (2003). *Physics Letters A* **315**, 23.
Gora, T. and Williams, F. (1969). *Physical Review* **177**, 1179.
Gradshtein, I. S. and Ryzhik, I. M. (1965). *Table of Integrals, Series, and Products*, Academic Press, New York.
Guertin, R. and Wilson, T. L. (1977). *Physical Review D* **15**, 1518.
Harrison, P. (2000). *Quantum Wells, Wires and Dots*, Wiley, New York.
Kemmer, N. (1939). *Proceedings of the Royal Society of London, Series A (Mathematical and Physical Sciences)* **173**, 91.
Lévy-Leblond, J.-M. (1992). *European Journal of Physics* **13**, 215.
Lévy-Leblond, J.-M. (1995). *Physical Review A (Atomic, Molecular, and Optical Physics)* **52**, 1845.
Lunardi, J. T., Pimentel, B. M., Teixeira, R. G., and Valverde, J. S. (2000). *Physics Letters A* **268**, 165.
Merad, M., Chetouani, L., and Bounames, A. (2000). *Physics Letters A* **267**, 225.
Morrow, R. A. (1987). *Physical Review B (Condensed Matter)* **35**, 8074.
Mustafa, O. and Mazharimousavi, S. H. (2006a). *Physics Letters A* **358**, 259.
Mustafa, O. and Mazharimousavi, S. H. (2006b). *Journal of Physics A (Mathematical and General)* **39**, 10537.
Mustafa, O. (2003). *Journal of Physics A (Mathematical and General)* **36**, 5067.
Nowakowski, M. (1998). *Physics Letters A* **244**, 329.
Peng, X. L., Liu, J. Y., and Jia, C. S. (2006). *Physics Letters A* **352**, 478.
Petiau, G., *Académie Royale de Belgique. Classe des Sciences. Mémoires. Collection* **16**.
Quesne, C. and Tkachuk, V. M. (2004). *Journal of Physics A (Mathematical and General)* **37**, 4267.
Quesne, C. (2006). *Annals of Physics 321*, 1221.
Rakityansky, S. A. (2004). *Physical Review B (Condensed Matter)* **70**, 205.
Serra, L. and Lipparini, E. (1997). *Europhysics Letters* **40**, 667.
Shi, J. J., Zhu, X. Q., Liu, Z. X., and Pan, S. H. (1997). *Physical Review B (Condensed Matter)* **55**, 4670.
Smagley, V. A., Mojahedi, M., and Osiñski, M. (2002). *Proceedings of the SPIE* **4646**, 258.
Snow, C. (1952). *Hypergeometric and Legendre Functions With Applications to Integral Equations of Potential Theory, National Bureau of Standards Applied Mathematics Series*, Vol. 19.
Tanaka, T. (2006). *Journal of Physics A (Mathematical and General)* **39**, 219.
Vakarchuk, I. O. (2005). *Journal of Physics A (Mathematical and General)* **38**, 4727.
Von Roos, O. (1983). *Physical Review B (Condensed Matter)* **27**, 7547.
Yi, L. Z., Diao, Y. F., Liu, J. Y., and Jia, C. S. (2004). *Physics Letters A* **333**, 212.
Zheng, Y., Lü, T., Liu, J., and Su, W. (1997). *Semiconductors Science and Techniques* **12**, 1235.